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A NOTE ON FIXED POINT THEOREMS OF NEW ITERATIVE SCHEME IN BANACH SPACES

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Abstract

In this paper, we have tried to established the stability and strong convergence results using new iterative scheme in Banach spaces under contractive-like mappings. Moreover, we will also show the example that the new iterative scheme converges faster that to another three-step iterative schemes.

1. Introduction

Fixed point theory plays an important role in applications of many branches of mathematics and the convergence results of Zamfirescu operators using Mann and Ishikawa iterative schemes introduced by Rhoades [26] in 1976 Berinde [3] established the class of operators that is more elaborate than the class Zamfirescu operators. After strong

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convergence of two-step iterative processes, in 2006, Rafiq studied the convergence of quasi-contractive mappings by a three-step iterative scheme. Olatinwo & Imoru [17] introduces convergence results of the class of generalized Zamfirescu operators under the Jungck-Ishikawa and the Jungck-Mann iteration scheme in and Olatinwo [19] studied the convergence for generalized Zamfirescu operation by Jongck- Noor iterative scheme Bosede [5] introduced strong convergence results of contractivelike mapping with the Jungck Ishikawa and the Jungck-Mann iterative scheme and Chugh, Renu and Kumar, Vivek [9, 10] studied of the strong convergence and stability result for the Jungck-SP and Jungck-Agrawal et al. iteration procedure in 2011. In the second case of this work we have the concept of fixed point of stability results of iteration methods. The stability of fixed point iteration methods was First studied by Ostrowaki [22], where they were the stability result of the Picard iteration method by Banach's contraction operator. After many researchers has studied this concept in various ways. Many researchers studying stability the following are Harder and Hicks [11], Rhoades [26], Berinde [4], Olatinwo [18] Osilike and Udomene [21] Osilike [20], and Singh et al. [27] Bosede and Rhoades [6] and Chugh Renu and Kumar Vivek [8, 9, 10]. The final concept of this task will study the intensity of iterative methods. The results obtained in this paper extend and improve the resluts in the existing results.

2. Preliminaries and Definitions

First some important definitions and theorems, which are useful for our main results. Let K be a closed convex subset of a Banach spaces B and let $T : K \to K$ be a selfmapping of K. Suppose that $F_T = \{q \in K : Tq = q\}$ is the set of fixed points of T. In Picard iterative scheme $u_0 \in K$ and $\{u\}_{s=0}^{\infty}$ defined by

$$u_{s+1} = Tu_s, \quad s = 0, 1, 2, \cdots \tag{1}$$

i.e. $u_{s+1} = f(T, u_s), \quad s = 0, 1, 2, \cdots$ and it is used to approximate the fixed points of operators satisfying the inequality

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \;\; \forall \;\; x, y \in K \;\; \& \;\; \alpha \in K[0, 1). \tag{2}$$

The equation (2) is called the Banach's contraction operator and also called Banach's of contractive Principle [2].

In 1953, W. R. Mann [15] introduced the following iterative scheme, for $u_0 \in K$ and $\{u_s\}_{s=0}^{\infty}$ defined by

$$u_{s+1} = (1-\alpha_s)u_s + \alpha_s T u_s \tag{3}$$

where $s \in N$, $\{\alpha_3\}$ is sequence of positive numbers in $[0,1]$ and it is called Mann iterative schemes.

In 1974, S. Ishikawa [12] introduced the following iterative scheme, for $u_0 \in K$ and $\{u_s\}_{s=0}^{\infty}$ defined by

$$u_{s+1} = (1-\alpha_s)u_s + \alpha_s T v_s \tag{4}$$

$$Vs = (1-\beta_s)u_s + \beta_s T u_s$$

where $s \in N$, $\{\alpha_s\}$, (β_s) are sequence of positive numbers in $[0,1]$ and it is called Ishikawa iterative schemes.

In 2000 Noor [16] introduced the following iterative scheme, for $u_0 \in K$ and $\{u_s\}_{s=0}^{\infty}$ defined by

$$\begin{aligned} u_{s+1} &= (1-\alpha_s)u_s + \alpha_s T v_s \\ v_s &= (1-\beta_s)u_s + \beta_s T w_s \\ w_s &= (1-\gamma_s)u_s + \gamma_s T u_s \end{aligned} \tag{5}$$

where $s \in N$, $\{\alpha_s\}$, (β_s), $\{\gamma_s\}$ are sequence of positive numbers in $[0,1]$ and it is called Noor iterative scheme.

In 2014, Abbas [1] introduced the following iterative scheme, for $u_0 \in K$ and $\{u_s\}_{s=0}^{\infty}$ defined by

$$\begin{aligned} u_{s+1} &= (1-\alpha_s)T v_s + \alpha_s T v w_s \\ v_s &= (1-\beta_s)T u_s + \beta_s T w_s \\ w_s &= (1-\gamma_s)u_s + \gamma_s T u_s \end{aligned} \tag{6}$$

where $s \in N$, $\{\alpha_s\}$, $(\beta_s), \{\gamma_s\}$ are sequence of positive numbers in $[0,1]$ and it is called Abbas iterative scheme.

Theorem 2.1 : Let K be a non-empty closed convex subset of a Banach space B and let $T : K \to K$ be a mapping on K. Then the operator T is called Zamfirescu operator if and only if there exists real numbers $r \in (0,1)$, s.t. $\in (0,1/2)$ such that for each u and v in K at least one of the following condition holds :

(a) $\|Tu - Tv\| \leq r\|u - v\|$

(b) $\|Tu - Tv\| \leq s\{\|u - Tu\| + \|v - Tv\|\}$

(c) $\|Tu - Tv\| \leq t\{\|u - Tv\| + \|v - Tu\|\}$.

Then T has a unique fixed point q and the Picard iterative scheme $\{u_n\}$ defined by

$$u_{n+1} = Tu_n, \quad n = 0, 1, \cdots$$

converges to q for any arbitrary but fixed $u_0 \in K$.

In 2005, Berinde [3] discussed a new class of operators on metric space, Banach space and Hilbert space and it is given by

$$\|Tu - Tv\| \leq 2\delta\|u - Ty\| + \delta\|u - v\| \;\; \forall \;\; u, v \in K \;\; \text{and} \;\; \delta \in [0, 1). \tag{7}$$

In 1976, Jungck introduced iterative scheme [13] as Let T and S be an arbitrary mappings on any set C with values in K satisfy $T(C) \subseteq S(C)$, where $S(C)$ is a complete subspace of K. For $u_0 \in C$, let $\{Su_n\}_{n=0}^{\infty}$ follows as

$$Su_{n+1} = Tu_n, \quad n = 0, 1, 2, \cdots$$

$$\text{i.e.} \quad Su_{n+1} = f(T, u_n), \quad n = 0, 1, 2, \cdots \tag{8}$$

is called Picard iterative schemes.

In 2005, Singh et al [27] introduced iteration processes as follows, for $u_0 \in C$, let $\{Su_n\}_{n=0}^{\infty}$ defined as

$$Su_{s+1} = (1 - \alpha_s)Su_s + \alpha_s Tu_s \tag{9}$$

where $s \in N$, $\{\alpha_s\}$ is sequence of positive numbers in $[0, 1]$ and it is called Jungck-Mann iterative scheme.

In 2008, Olatinwo and Imoru [17] introduced iteration processes as follows, for $u_0 \in C$, let $\{Su_n\}_{n=0}^{\infty}$ defined as

$$Su_{s+1} = (1 - \alpha_s)Su_s + \alpha_s Tv_s \tag{10}$$

$$Sv_s = (1 - \beta_s)Su_s + \beta_s Tw_s$$

where $s \in N, \{\alpha_s\}, \{\beta_s\}$ are sequence of positive numbers in $[0, 1]$ and it is called Jungck-Ishikawa iterative scheme.

In 2008, Olatinwo [19] introduced iteration processes as follows, for $u_0 \in C$, let $\{Su_0\}_{n=0}^{\infty}$ defined as

$$Su_{s+1} = (1-\alpha_s)Su_s + \alpha_s Tv_s \tag{11}$$

$$Sv_s = (1-\beta_s)Su_s + \beta_s Tw_s$$

$$Sw_s = (1-\gamma_s)Su_s + \gamma_s Tu_s$$

where $s \in N, \{\alpha_s\}, \{\beta_s\}, \{\gamma_m\}$ are sequence of positive numbers in $[0,1]$ and it is called Noor iterative scheme.

Using a new idea, Osilike [20] considered the following contractive condition: there exist a real number $L \geq 0, \alpha \in [0,1]$ such that for each $u, v \in K$, we have

$$\|Tu - Tv\| \leq L\|u - Tu\| + \alpha\|u - v\|. \tag{12}$$

Imoru and Olatinwo [17] extended the results of oslike [20] using the following contractive condition : there exists a real number $\alpha \in [0,1])$ and a monotonic increasing function $\phi : R^+ \to R^+$ such that $\phi(0) = 0$ and $\forall \;\; u, vinK$, we have

$$\|Tu - Tv\| \leq \phi(\|u - Tv\|) + \alpha\|u - v\| \tag{13}$$

Jungck [13] studied that the mappings T and S satisfying

$$\|Tu - Tv\| \leq \alpha\|Su - Sv\|$$

$\forall \; u, v \in K, 0 \leq \alpha < 1$ has a unique common fixed point. For $C = K$ & $S = Id$ above operator becomes well know contraction mapping.

Olainwo [18] introduced an operator that is a generalization of the Zamfirescu operator, called contractive-like operators and defined as

(A) There exists a real number $M \geq 0, \alpha \in [0,1)$ and a monotonic increasing function $\phi : R^+ \to R^+$ such that $\phi(0) = 0$ and $\forall \; u, v \in K$, we have

$$\|Tu - Tv\| \leq \phi(\|Su - Tu\|) + \alpha\|Su - Sv\|$$

(B) There exists a real number $M \geq 0, \alpha \in [0,1)$ and a monotonic increasing function $\phi(R^+ \to R^+$ such that $\phi(0) = 0$ and $\forall \;\; u, v \in K$m we have

$$\|Tu - Tv\| \leq \frac{\phi(\|Su - Tu\|) + \alpha\|Su - Sv\|}{1 + M\|Su - SV\|}.$$

Bosede and Rhoades [6] introduced an operator that is a generalization of the Zamfirescu operator, called contractive-like operators and defined as : there exists a real number $M \geq 0, \delta in[0,1)$ and a monotonic increasing function $\phi : R^+ \to R^+$ such that $\phi(0) = 0$ and $\forall\ u, v \in K$ we have

$$\|Tu - Tv\| \leq e^{M\|Su-Tv\|}[\phi\{\|Su - Tv\|\} + \delta\|Su - Sv\|] \tag{14}$$

In this paper, we define following new New iteration process and is defined as follows

$$Su_{s+1} = (1 - \alpha_s)Tv_s + \alpha_s Tw_s$$

$$Sv_s = (1 - \beta_s)Tu_s + \beta_s Tw_s$$

$$Sw_s = (1 - \gamma_s)Su_s + \gamma_s Tu_s \tag{15}$$

where $s \in N, \{\alpha_s\}, \{\beta_s\}, \{\gamma_s\}$ are sequence of positive numbers in $[0, 1)$ and it is called New iterative scheme.

We are implying this contractive-like operators there exists a real number $M \geq 0$, $\alpha \in [0, 1)$ and a monotonic increasing function $\phi :?R^+ \to K^+$ such that $\phi(0) = 0$ and $\forall\ u, v \in K$, we have

$$\|Tu - Tv\| \leq e^{M\|Su-Tv\|}[\phi\{\|Su - Tv\|\} + \alpha\|Su - Sv\|]. \tag{16}$$

We shall need the following lemma and definition for our results:

Lemma 2.1 [3] : Let p be a real number such that $0 \leq p < 1$ and $\{\epsilon_s\}_{s=0}^{\infty}$ be a sequence of non-negative numbers such that $\lim_{s\to\infty} \epsilon_s = 0$, then for any sequence $\{u_s\}_{n=0}^{\infty}$ is satisfying $u_{s+1} \leq \delta u_s + \epsilon_s,\ s = 0, 1, 2, \cdots$, we have $\lim_{s\to\infty} u_s = 0$.

Definition 2.2 : Let $T, S : C \to K$ be non-self mapping on C satisfying $T(C) \subseteq S(C)$, where $S(C)$ is a complete subspace of K and S is an injective mapping. Let b be a coincidence point of T and S i.e. $Sb = Tb = q$ (say). Let, for $u_0 \in C$. The sequence $\{Su_m\}_{n=0}^{\infty}$ generated by the iteration scheme $Su_n = f(T, u_n).n \geq 0$ (16) will be called (S, T)-stable if and only if

$$\lim_{n\to\infty} \epsilon_n = 0 \Leftrightarrow \lim_{m\to\infty} Sv_n = q.$$

3. Main Results

3.1. Strong Convergence Results of Fixed Point Theorem in Banach Space

Theorem 3.1 : Let K be a non-empty closed convex subset of a Banach space B and let T, $S : K \to K$ be non-self mapping on an arbitrary set C such that $T(C) \subseteq S(C)$, where $S(C)$ is a complete subspace of K and S is an injective mapping. Let y be a coincidence point of T and S i.e. $Ty = Sy = q$. Suppose T, S are satisfy the condition (16) and for $u_0 \in C$, let $\{Su_m\}_{n=0}^{\infty}$ be a New iterative scheme defined by (15). Then the sequence $\{Su_m\}_{n=0}^{\infty}$ converges strongly to q.

Proof : Let y be the set of coincidence points of T and S. We use condition (16) to establish that T and S have a unique coincidence point y, i.e. $Sy = Ty = q$ (sav). Suppose that $\exists\ y_1, y_2 \in Y$ such that $Sy_1 = Ty_1 = q_1$ and $Sy_2 = Ty_2 = q_2$. If $q_1 = q_2$, then $Sy_1 = Sy_2$ and since S is an injective, it follows that $y_1 = y_2$.

If $q_1 \neq q_2$, then using contractive condition (16) we have

$$\begin{aligned} 0 &\leq \|q_1 - q_2\| = Ty_1 - Ty_2\| \\ &\leq e^{L\|Sy_1, Ty_2\|}\|\{\varphi(\|Sy_1 - Ty_2\|) + \alpha\|Sy_1 - Ty_2\|\} \\ &= \alpha\|q_1 - q_2\| \end{aligned}$$

Which leads to $(1-\alpha)\|q_1 - q_2\| \leq 0$, from which it follows that $(1-\alpha) > 0$ since $\alpha \in [0, 1)$ but $\|q_1 - q_2\| \leq 0$, since norm is non-negative, which is a contradiction. Therefore, we have that $\|q_1 - q_2\| = 0$ i.e. $q_1 - q_2 = q$. Since $q_1 = q_2$, then we have that $q_1 = Sy_1 = Ty_1 = Sy_2 = Ty_2 = q_2$ leading to $Sy_1 = Sy_2 \Rightarrow y_1 = y_2 = y$ (since S is an injective). Hence y is a unique coincidence point of T and S.

We now prove that $(Su_m\}_{m=0}^{\infty}$ converges strongly to q. Therefore, from (15) and (16) we have

$$\begin{aligned} \|Su_{m+1} - q\| &= (1-\alpha_m)Tv_m + \alpha_m Tw_m - q\| \\ &\leq (1-\alpha_m)\|Tv_m - q\| + \alpha_m\|Tw_m - q\| \\ &= (1-\alpha_m)\|Tv_m - Ty\| + \alpha_m\|Tw_m - Ty\| \\ &= (1-\alpha_m)\|Tv - Tv_m\| + \alpha_m\|Ty - Tw_m\| \end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_m)e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|)+\alpha\|Sy-Sv_m\|\} \\
&\quad +\alpha_m e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|)+\alpha\|Sy-Sw_m\|\} \\
&= (1-\alpha_m)\alpha\|Sy-Sv_m\|+\alpha_m\alpha\|Sy-Sw_m\| \\
&= (1-\alpha_m)\alpha\|Sv_m-q\|+\alpha_m\alpha\|Sw_m-q\|. \qquad (17)
\end{aligned}
$$

Now, we have the following estimates :

$$
\begin{aligned}
\|Sv_m-q\| &= \|(1-\beta_m)Tu_m+\beta_m Tw_m-q\| \\
&\le (1-\beta_m)\|Tu_m-q\|+\beta_m\|Tw_m-q\| \\
&= (1-b_m)\|Tu_m-Ty\|+\beta_m\|Tw_m-Ty\| \\
&= (1-\beta_m)\|Ty-Tu_m\|+\beta_m\|Ty-Tw_m\| \\
&\le (1-\beta_m)e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|)+\alpha\|Sy-Su_m\|\} \\
&\quad +\beta_m e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|)+\alpha\|Sy-Sw_m\|\} \\
&= (1-\beta_m)\alpha\|Sy-Su_m\|+\beta_m\alpha\|Sy-Sw_m\| \\
&= (1-\beta_m\alpha\|q-Su_m\|+\beta_m\alpha\|q-Sw_m\| \\
&= (1-\beta_m)\alpha\|Su_m-q\|+\beta_m\alpha\|Sw_m-q\| \qquad (18)
\end{aligned}
$$

and

$$
\begin{aligned}
\|Sw_m-q\| &= \|(1-\gamma_m)Su_m+\gamma_m Tu_m-q\| \\
&\le (1-\gamma_m)\|Su_m-q\|+\gamma_m\|Tu_m-q\| \\
&= (1-\gamma_m)\|Su_m-q\|+\gamma_m\|Tu_m-Ty\| \\
&= (1-\gamma_m)\|Su_m-q\|+\gamma_m\|Ty-Tu_m\| \\
&\le (1-\gamma_m)\|Su_m-q\| \\
&\quad +\gamma_m e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|)+\alpha\|Sy-Su_m\|\} \\
&= (1-\gamma_m)\alpha\|Su_m-q\|+\gamma_m\alpha\|Sy-Sw_m\| \\
&= (1-\gamma_m\alpha\|Su_m-q\|+\gamma_m\alpha\|Su_m-q\| \\
&= [1-\gamma_m(1-\alpha)]\|Su_m-q\|. \qquad (19)
\end{aligned}
$$

Putting the value of (19) in equation (18), we get

$$
\begin{aligned}
\|Sv_m-q\| &\le \alpha(1-\beta_m)\|Su_m-q\|+\alpha\beta_m[1-\gamma_m(1-\alpha)]\|Su_m-q\| \\
&= \alpha[-\beta_m\gamma_m(1-\alpha)]\|Su_m-q\|. \qquad (20)
\end{aligned}
$$

Putting the value of (19) and (20) in equation (17), we get

$$\begin{aligned}
\|Su_{m-1} - q\| &\leq \alpha^2(1-\alpha_m)[1-\beta_m\gamma_m(1-\alpha)]\|Su_m - q\| \\
&\quad +\alpha_m\alpha[1-\gamma_m(1-\alpha)]\|Su_m - q\| \\
&= \alpha[\alpha(1-\alpha_m)\{1-\beta_m\gamma_m(1-\alpha)\} \\
&\quad +\alpha_m\{1-\gamma_m(1-\alpha)\}]\|Su_m - q\| \\
&\leq [\alpha(1-\alpha_m)\{1-\beta_m\gamma_m(1-\alpha)\} \\
&\quad +\alpha_m\{1-\gamma_m(1-\alpha)\}]\|Su_m - q\| \\
&\leq [(1-\alpha_m)\{1-\gamma_m(1-\alpha)\} \\
&\quad +\alpha_m\{1-\gamma_m(1-\alpha)\}]\|Su_m - q\| \\
&\leq [1-\gamma_m(1-\alpha)]\|Su_m - q\| \\
&\leq \prod_{i=0}^{m}[1-\gamma_i(1-\alpha)] \\
&\leq e^{(1-\alpha)\sum_{i=0}^{m}\gamma_i}\|u_0 - q\| \qquad (21)
\end{aligned}$$

Since $0 \leq \alpha < 1, \gamma_i \in [0,1]$ and $\sum_{m=0}^{\infty}\gamma_m = \infty$, so $e^{-(1-\alpha)\sum_{i=0}^{m}\gamma_i} \to 0$ as $m \to \infty$. Hence, it follows from (21) that $\lim_{m\to\infty}\|Su_{m+1} - q\| = 0$. Therefore $\{Su_m\}_{n=0}^{\infty}$ converges strongly to q.

3.2. Stability Results of Fixed Point Theorem in Banach Spaces

Theorem 3.2 : Let K be a non-empty closed convex subset of a Banach Spaces B and let $T, S : K \to K$ be non-self mappings on an arbitrary set C such that $T(C) \subseteq S(C)$, where $S(C)$ is a complete subspace of K and S is an injective mapping. Let y be a coincidence point of T and S i.e. $Ty = Sy = q$ Suppose T, S are satisfying the condition (16) and for $u_0 \in C$, let $\{Su_m\}_{n=0}^{\infty}$ be New iterative scheme (15) converging to q. Then the New iterative scheme (S, T) - stable.

Proof : Suppose $\{Sv_m\}_{m=0}^{\infty} \subset K$ be an arbitrary sequence and define

$$\epsilon_m = \|Sv_{m+1} - (1-\alpha_m)Tb_m - \alpha_m Tc_m\|$$

where $Sb_m = (1-\beta_m)Tv_m + \beta_m Tc_m, Sc_m = (1-\gamma_m)Sv_m + \gamma_m Tv_m$ and let $\lim_{m\to\infty}\epsilon_m = 0$. Now we have to prove that $\lim_{m\to\infty} Sv_m = q$.

Then it follows from (15) and (16) that

$$\begin{aligned}
\|Sv_{m+1} - q\| &\leq \|Sv_{m+1} - (1-\alpha_m)Tb_m - \alpha_m Tc_m\| \\
&\quad + \|(1-\alpha_m)Tb_m + \alpha_m Tc_m - q\| \\
&\leq \epsilon_m + (1-\alpha_m)\|Tb_m - q\| + \alpha_m\|Tc_m - q\| \\
&= (1-\alpha_m)\|Tb_m - Ty\| + \alpha_m\|Tc_m - Ty\| + \epsilon_m \\
&= (1-\alpha_m)\|Ty - Tb_m\| + \alpha_m\|Ty - Tc_m\| + \epsilon_m \\
&\leq (1-\alpha_m)^{eL\|Sy-Ty\|}|\{\varphi(\|Sy - Ty\|) + \alpha\|Sy - Sb_m\|\} + \epsilon_m \\
&\quad + \alpha_m^{eL\|Sy-Ty\|}\{\varphi(\|Sy - Ty\|) + \alpha\|Sy - Sc_m\|\} + \epsilon_m \\
&= (1-\alpha_m)\alpha\|Sy - Sb_m\| + \alpha_m a\|Sy - Sc_m\| + \epsilon_m \\
&= (1-\alpha_m)\alpha\|Sb_m - q\| + \alpha_m a\|sc_m - q\| + \epsilon_m. \qquad (22)
\end{aligned}$$

Now, we have the following estimates

$$\begin{aligned}
\|Sb_m - q\| &= \|(1-\beta_m)Tv_m + \beta_m Tc_m - q\| \\
&\leq (1-\beta_m)\|Tv_m - q\| + \beta_m\|Tc_m - q\| \\
&= (1-\beta_m)\|Tv_m - Ty\| + \beta_m\|Tc_m - Ty\| \\
&= (1-b_m)\|Ty - Tv_m\| + \beta_m\|Ty - Tc_m\| \\
&\leq (1-\beta_m)^{eL\|Sy-Ty\|}\{\varphi(\|Sy - Ty\| + \alpha\|Sy - Sv_m\|\} \\
&\quad + \beta_m^{eL\|Sy-Ty\|}\{\varphi(\|Sy - Ty\| + \alpha\|Sy - Sc_m\|\} \\
&= (1-\beta_m)\alpha\|Sy - Sv_m\| + \beta_m\alpha(\|Sy - Sc_m\|) \\
&= (1-\beta_m)\alpha\|q - Sv_m\| + \beta_m\alpha(\|q - Sc_m\|) \\
&= (1-\beta_m)\alpha\|Sv_m - q\| + \beta_m\alpha(\|Sc_m - q\|) \qquad (23)
\end{aligned}$$

and

$$\begin{aligned}
{}_m - q\| &= \|(1-\gamma_m)\|Sv_m + \gamma_m Tv_m - q\| \\
&\leq (1-\gamma_m)\|Sv_m - q\| + \gamma_m\|Tv_m - q\| \\
&= (1-\gamma_m)\|Sv_m - q\| + \gamma_m\|Tv_m - Ty\| \\
&= (1-\gamma_m)\|Sv_m - q\| + \gamma_m\|Ty - Tv_m\|
\end{aligned}$$

$$\begin{aligned} &\leq (1-\gamma_m)\|Sv_m - q\| + \gamma_m^{eL\|Sy-Ty\|}\|\varphi(\|Sy-Ty\|) + \alpha\|Sy - Sv_m\|\} \\ &= (1-\gamma_m)\|Sv_m - q\| + \gamma_m^a\|Sy - Sv_m\| \\ &= (1-\gamma_m)\|Sv_m - q\| + \gamma_m^a\|Sv_m - q\|\|Sv_m - q\|. \end{aligned} \tag{24}$$

Putting the value of (24) in equation (23), we get

$$\begin{aligned} \|Sb_m - q\| &\leq \alpha(1-\beta_m)\|Sv_m - q\| + \alpha\beta_m[1-\gamma_m(1-\alpha)]\|Sv_m - q\| \\ &= \alpha[1-\beta_m\gamma_m(1-\alpha)]\|Sv_m - q\| \end{aligned} \tag{25}$$

Putting the value of (25) and (24) in equation (22), we get

$$\begin{aligned} \|Sv_{m+1} - q\| &\leq \alpha^2(1-\alpha_m)[1-\beta_m\gamma_m(1-\alpha)]\|Sv[\alpha^2(1-\alpha)]\|Sv_m - q\| \\ &\quad +\alpha_m a[1-\gamma_m(1-\alpha)]\|Sv_m - q\| + \epsilon_m \\ &= [\alpha^2(1-\alpha_m)\{1-\beta_m\gamma_m(1-\alpha)\} \\ &\quad +a\alpha_m\{1-\gamma_m(1-\alpha)\}]\|Sv_m - q\| + \epsilon_m \\ &= h\|Sv_m - q\| + \epsilon_m \end{aligned} \tag{26}$$

where,

$$h = (\alpha^2(1-\alpha_m)\{1-\beta_m\gamma_m(1-\alpha)\} + a\alpha_m\{1-\gamma_m(1-\alpha)\}.$$

Taking the limit of both sides of above equation as $m \to \infty$

$$\lim_{m\to\infty} \|Sv_{m+1} - q\| \leq h \lim_{m\to\infty} \|Sv_m - q\| + \lim_{m\to\infty} \epsilon_m.$$

But $\lim_{m\to\infty} \epsilon_m = 0$. Therefore,

$$\lim_{m\to\infty} \|Sv_{m+1} - q\| \leq h \lim_{m\to\infty} \|Sv_m - q\|. \tag{27}$$

Now, using $0 < A \leq \alpha_m$ and $\alpha \in [0,1)$, we have

$$\begin{aligned} h &= a[a(1-\alpha_m)\{1-\beta_m\gamma_m(1-\alpha)\} + \alpha_m\{1-\gamma_m(1-\alpha)\}] \\ &\leq [a(1-\alpha_m)\{1-\beta_m\gamma_m(1-\alpha)\} + \alpha_m\{1-\gamma_m(1-\alpha)\}] \\ &\leq [(1-\alpha_m)\{1-\gamma_m(1-\alpha)\} + \alpha_m\{1-\gamma_m(1-\alpha)\}] \\ &\leq [1-\gamma_m(1-\alpha)] < 1. \end{aligned}$$

Therefore,

$$h = [a^2(1-\alpha_m)\{1\beta_m\gamma_m(1-\alpha)\} + \alpha\alpha_m\{1-\gamma_m(1-\alpha)\}]. \tag{28}$$

Using (27), (28) and lemma (2.1), we have

$$\lim_{m\to\infty} Sv_m = p.$$

Conversely, let $\lim_{m\to\infty} Sv_m = q$.
Then using contractive condition (16) and the triangle inequality, we have

$$\begin{aligned}
\epsilon_m &= \|Sv_{m+1} - (1-\alpha_m)Tb_m - \alpha_m Tc_m\| \\
&\leq \|Sv_{m+1} - q\| + (1-\alpha_m)\|q - Tb_m\| + \alpha_m t\|q - Tc_m\| \\
&= \|Sv_{m+1} - q\| + (1-\alpha_m)\|Ty - Tb_m\| + \alpha_m\|Ty - Tc_m\| \\
&\leq \|Sv_{m-1} - q\| + (1-\alpha_m)^{eL\|Sy-Ty\|}\{\varphi(\|Sy - Ty\|) + \alpha\|Sy - Sb_m\|\} \\
&\quad +\alpha_m^{eL\|Sy-Ty\|} + \{\varphi(\|Sy - Ty\|) + \alpha\|Sy - Sc_m\|\} \\
&= \|Sv_{m+1} - q\| + (1-\alpha_m)\alpha\|Sy - Sb_m\| + \alpha_m^a\|Sy - Sc_m\| \\
&= \|Sv_{m+1} - q\| + (1-\alpha_m)\alpha\|Sb_n - q\| + \alpha_m^a\|Sc_n - q\|. \qquad (29)
\end{aligned}$$

Putting the value of equations (25) and (24) in equation (29), we have

$$\begin{aligned}
\epsilon_m &\leq \|Sv_{m+1} - 1\| + \alpha^2(1-\alpha_m)[1 - \beta_m\gamma_m(1-\alpha)]\|Sv_m - q\| \\
&\quad +\alpha_m\alpha[1 - \gamma_m(1-\alpha)]\|Sv_m - q\| \\
&\quad +\|Sv_{m+1} - q\| + [\alpha^2(1-\alpha_m)\{1 - \beta_m\gamma_m(1-\alpha)\} \\
&\quad +\alpha\alpha_m\{1 - \gamma_m(1-\alpha)\}]\|Sv_m - q\| \\
&\quad \|Sv_{m+1} - q\| + \delta\|Sv_m - q\| \qquad (30)
\end{aligned}$$

where

$$\delta = [\alpha^2(1-\alpha_m)\{1 - \beta_m\gamma_m(1-\alpha)\} + a\alpha_m\{1 - \gamma_m(1-\alpha)\}].$$

Taking the limit of both sides of above equation as $m \to \infty$

$$\lim_{m\to\infty} \epsilon_m \leq \lim_{m\to\infty} \epsilon_m\|Sv_{m+1} - q\| + h \lim_{m\to\infty} \|Sv_m - q\|.$$

But $\lim_{m\to\infty} Sv_m = q$. Therefore, $\lim_{m\to\infty} \epsilon_m = 0$.
Hence the New iterative scheme is (S, T) - stable.

Example 3.3 : We solve the following equation

$$x^2 - 7x - 10 = 0 \qquad (31)$$

and prove that New iteration converges faster than Jungck-Noor, jungck-Ishikawa and Jungck-Mann iterative schemes. We solve the equation by rewriting as

$$Sx = 7x \ \ \& \ \ Tx = x^2 - 10 \tag{32}$$

To solve equation (31) using simple iterative schemes (Abbas, Noor, Ishikawa and Mann), we write this equation as follows :

$$x = Tx = (x^2 - 10)/7.$$

Then the coincidence point of S & T in (32) leads to the solution of equation (31). We show our output in the table 1 and table 2 by taking initial approximation $x_0 = 1$ and $\alpha_n = \beta_n = \gamma_n = 0.9$.

3.4. Conclusion : Following are the convergence events of Jungck type iterative processes in table - 1: Jungck-Mann, Jungck-Ishikawa, Jungck-Noor and New iterative schemes while table - 2 follows the convergence events of simple iteration schemes Mann, Ishikawa, Noor and Abbas. After looking at both the tables, we come to the conclusion that the result of table -2 is faster than results of table -1.

TABLE 1

No.	Jungck-Mann Iterative Scheme			Jungck-Ishikawa Iterative Scheme			Jungck-Noor Iterative Scheme			New Iterative Scheme		
n	Su_{m+1}	Tu_m	u_{n+1}	Su_{m+1}	Tu_m	u_{n+1}	Su_{m+1}	Tu_m	u_{n+1}	Su_{m+1}	Tu_m	u_{n+1}
0	-7.4	-9	-1.05714	-7.2942	-9	-1.04203	-7.32276	-9	- 1.04611	-8,83276	-9	-1.26182
1	-8.7342	-8.88245	-1.24774	-8,32247	-8.91418	-1.18892	-8.45874	-8.90566	-1.20839	-8.53997	-8.4078	-1.22
2	-8.47224	-8,44314	-1.21032	-8.48639	-8.58646	-1.21234	-8.51605	-8.53979	-1.21658	-8.52031	-8.51161	-1.21719
3	-8.52884	-8.53512	-1.21841	-8.51352	-8.53023	-1.21622	-8.5188	-8.51994	-1.21697	-8.51902	-8.51845	-1.217
4	-8.51682	-8.51549	-1.21699	-8.51803	-8.52082	-1.21686	-8.51893	-8.51898	-1.21699	-8.51894	-8.5189	-1.21699
5	-8.51938	-8.51967	-1.21705	-8.51878	-8.51925	-1.21697	-8.51893	-8.51894	-1.21699	-8.51893	-8.51893	-1.21699
6	-8.51884	-8.51878	-1.21698	-8.51891	-8.51899	-1.21699	-8.51893	-8.51893	-1.21699	-8.51893	-8.51893	-1.21699
7	-8.51895	-8.51897	-1.21699	-8.51893	-8.51894	-1.21699	-8.51893	-8.51893	-1.21699	-8.51893	-8.51893	- 1.21699
8	-8.51893	-8.51893	-1.21699	-8.51893	-8.51894	-1.21699	-8.51893	-8.51893	-1.21699	-8.51893	-8.51893	-1.21699
9	-8.51893	-8.51893	-1.21699	-8.51893	-8.51893	-1.21699	- 8.51893	-8.51893	-1.21699	-8.51893	-8.51893	-1.21699
10	-8.51893	-8.51893	-1.21699	-8.51893	-8.51893	-1.21699	-8.51893	-8.51893	-1.21699	-8.51893	-8.51893	-1.21699

TABLE 2

No.	Mann Iterative Scheme		Ishikawa Iterative Scheme		Noor Iterative Scheme		Abbas Iterative Scheme	
n	Tu_m	u_{m+1}	Tu_m	u_{m+1}	Tu_m	u_{m+1}	Tu_m	u_{m+1}
0	-1.28571	-1.05714	-1.28571	-1.04203	- 1.28571	-1.04611	-1.28571	-1.26182
1	-1.26892	-1.24774	-1.27345	-1.18892	-1.27224	-1.20839	-1.20111	-1.22
2	-1.20616	-1.21032	-1.22664	-1.21234	-1.21997	-1.21658	-1.21594	-1.21719
3	-1.2193	-1.21841	-1.2186	-1.21622	-1.21713	-1.21697	-1.21692	-1.217
4	-1.2165	-1.21669	-1.21726	-1.21686	-1.217	-1.21699	-1.21699	-1.21699
5	- 1.2171	-1.21705	-1.21704	-1.21697	-1.21699	-1.21699	-1.21699	-1.21699
6	-1.21697	-1.21698	-1.217	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699
7	-1.217	-1.21699	-1.21699	- 1.21699	-1.21699	-1.21699	- 1.21699	-1.21699
8	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21609	-1.21699
9	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21690	-121699
10	-1.21699	-1.21699	-1.21.699	-1.21699	1.21699	- 1.21699	-1.21699	-1.21600

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